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Introduction to a Brownian Quasiparticle Model

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A description intermediate between the usual stochastic description of a Brownian particle and the deterministic description of a classical particle is proposed. It is based on a model which utilizes the notions of a current velocity and of an osmotic velocity, and which generates a random process which allows us to associate with any given initial and final conditions a unique differentiable trajectory. This intermediate description of the Brownian motion, in terms of quasiparticles with quasideterministic behavior, gives back the same mean and the same variance as does the usual stochastic description.

KEY WORDS: Brownian motion; Fokker–Planck equation; current velocity; osmotic velocity; quasideterministic behavior; quasiparticles.

1. INTRODUCTION

Before exposing the problem we are going to handle, let us summarize some results of stochastic physics. Let $X(t)$ be a process in $\mathbb{R}^n$ which satisfies a generalized Langevin equation of the form

$$dX(t) = D(X(t), t)\, dt + dW(t)$$

where $X(t)$ is a vector of $\mathbb{R}^n$, D a regular field of vectors, dt a macroscopically infinitesimal increment of time, and $W(t)$ a Wiener process, i.e., a process whose increments are Gaussian, with zero mean, and whose variance matrix $|t - s|[\sigma_{ij}^2]$, $i, j = 1,..., n$, is proportional to the time increment $|t - s|$. We call the matrix of diffusion the matrix $[M] = \frac{1}{2}[\sigma_{ij}^2]$. Given D, the probability density in $\mathbb{R}^n$, $f(X, t)$, satisfies the Fokker–Planck equation

$$\frac{\partial}{\partial t} f(X, t) = -\operatorname{div}_X(f(X, t)D(X, t)) + \operatorname{div}_X([M] \operatorname{grad}_X f(X, t)) \qquad (1)$$

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The conditional probability density $P(Xt|X_0t_0)$ that the variable $X(t)$ takes the value X at time t, knowing that it has the value X_0 at time t_0, is the fundamental solution $G(X, t, X_0, t_0)$ of this equation, i.e., the solution that tends to $\delta(X - X_0)$ when t tends to zero ($t > t_0$).

In this paper we limit ourselves to a D field of vectors independent of time and linear in X. In this case, we can explicitly integrate the differential deterministic system

$$dX_t^D = D(X_t^D)\, dt$$

and we call $X_t^D(t - t_0, X_0)$ the solution that takes the value X_0 at time t_0. The conditional probability density is then a function of $t - t_0$, and we write it as $f^\delta(X, t - t_0, X_0)$:

$$P(Xt|X_0t_0) = f^\delta(X, t - t_0, X_0), \qquad t > t_0$$

Finally, the mean of the process is equal to X_t^D,

$$E\{X(t)\} = \int Xf^\delta(X, t - t_0, X_0)\, dX = X_t^D(t - t_0, X_0)$$

and the elements of its variance matrix are functions of $t - t_0$:

$$E\{(X_i(t) - X_{t,i}^D)(X_j(t) - X_{t,j}^D)\} = C_{ij}(t - t_0)$$

We are now in a position to expose the problem that is the object of this paper. Let us consider a Brownian particle submitted to a double initial and final condition: to leave X_0 at time t_0 and to arrive at X' at time t'. To this double condition there correspond an infinity of possible stochastic trajectories (nondifferentiable). Our aim is to build a model of quasiparticles which involves a sole differentiable trajectory, going from (X_0t_0) to $(X't')$, and representing a certain mean behavior of the Brownian particle between those two points. This mean behavior must be such as to give back, by integrating over the final position X', the mean $X_t^D(t - t_0, X_0)$ and the variance matrix $[C_{ij}(t - t_0)]$ of the process $X(t)$ at time t, given the initial (X_0t_0) condition. Moreover, it must be the same thing for any initial $I(X_0)$ density. However, we do not ourselves impose any condition on the covariance at two different instants.

2. DESCRIPTION OF THE MODEL

The first idea which comes in mind is to take as a trajectory the partial mean of the process $X(t)$, which is calculated with the conditional probability density $P(Xt|X_0t_0, X't')$ to be at (Xt), knowing that we are at X_0 at time t_0 and at X' at time t' ($t' > t > t_0$),

$$Y(t, X', t', X_0, t_0) = \int XP(Xt|X_0t_0, X't')\, dX$$

We easily check that the mean of Y over the final condition X', which is calculated with the probability density $P(X't'|X_0t_0)$, is independent of t' and is equal to the mean X_t^D of the process $X(t)$. But we can check in particular cases (we handle in the appendix the simplest case, $n = 1$, $D = 0$) that the variance matrix of Y is not equal to the variance matrix of $X(t)$. Thus, the trajectory Y does not satisfy the criteria we have chosen. We now present a model which satisfies these criteria.

Our model is founded on the notions of the current velocity and of the osmotic velocity associated with the process $X(t)$ for the initial (X_0t_0) condition.[1] The current velocity $V(X, t - t_0, X_0)$ and the osmotic velocity $\Omega(X, t - t_0, X_0)$ are respectively the half sum and the half difference of the mean velocity departing from $X(t)$ (mean forward velocity) and of the mean velocity arriving at $X(t)$ (mean backward velocity). Therefore V and Ω are linked by the relation

$$V(X, t - t_0, X_0) = D(X) - \Omega(X, t - t_0, X_0)$$

Moreover, Ω is a function of the probability density

$$\Omega(X, t - t_0, X_0) = [M] \operatorname{grad}_X \log f^\delta(X, t - t_0, X_0)$$

and V satisfies the continuity equation in the Eulerian description.

Our idea introduces the preceding notions into a Lagrangian description. We consequently postulate a continuous dynamic evolution, which is described by the variable X_t,

$$\frac{d}{dt} X_t = V(X_t, t - t_0, X_0) = D(X_t) - [M] \operatorname{grad}_{X_t} \log f^\delta(X_t, t - t_0, X_0) \quad (2)$$

and which satisfies the continuity equation along the stream:

$$\frac{\partial}{\partial t} f^\delta(X_t, t - t_0, X_0) = -\operatorname{div}_{X_t}(f^\delta(X_t, t - t_0, X_0) V(X_t, t - t_0, X_0)) \quad (3)$$

We verify that the compatibility equation between (2) and (3) is just the Fokker–Planck equation (1) written for $X = X_t$.

In various particular cases, we have developed this model in the following way:

(a) We solve Eq. (1); for a field D independent of time and linear in X, the fundamental solution f^δ is a Gaussian function whose argument is a quadratic form of $(X - X_t^D)$, where $X_t^D(t - t_0, X_0)$ is the solution of the deterministic system $dX_t^D = D(X_t^D)\, dt$.

(b) We put this solution f^δ in (2), which thus becomes an explicit differential system, which we integrate, thus obtaining the general solution

$$X_t(t - t_0, \Lambda, X_0) = X_t^D(t - t_0, X_0) + [\phi(t - t_0)]\Lambda$$

where the vector Λ represents the n arbitrary constants of integration, and where $[\phi(t - t_0)]$ is an $n \times n$ matrix which vanishes for $t = t_0$.

(c) It appears that the Jacobian $J_{t'} = \mathscr{D}(X_{t'})/\mathscr{D}(\Lambda) = \det[\phi(t' - t_0)]$ is different from zero for $t' > t_0$; given a final condition $X_{t'} = X'$ at time t', we thus determine the vector Λ. This gives us a well-defined trajectory X_t going from $(X_0 t_0)$ to $(X' t')$.

(d) We associate with this trajectory (and thus with the corresponding vector Λ) the probability density $f^\delta(X', t' - t_0, X_0)$ for arrival at $(X't')$ after leaving $(X_0 t_0)$. With this probability density, and by integration over X', it is then possible to calculate the mean and the variance matrix of X_t. These quantities, which are independent of t', are the same as those of the process $X(t)$. In fact, we use a law of distribution for Λ, such that

$$\rho(\Lambda, t' - t_0, X_0)\, d\Lambda = f^\delta(X', t' - t_0, X_0)\, dX'$$

and it appears that this law ρ is a function only of Λ, and that it has a Gaussian form. It is thus possible to introduce a random process $Z(t - t_0, \Lambda, X_0)$ in the following form:

$$\text{process } Z: \quad \begin{cases} Z(t - t_0, \Lambda, X_0) = X_t^D(t - t_0, X_0) + [\phi(t - t_0)]\Lambda \\ \text{Gaussian distribution law } \rho(\Lambda) \end{cases}$$

and we check that the mean and the variance matrix, respectively, of this process Z are equal to those of the process $X(t)$.

(e) When X_0 becomes a random variable, denoted Ξ, with the (normalized) initial density $I(\Xi)$, we easily verify that the mean and the variance matrix of the process Z are still the same as those of the process $X(t)$. Indeed, in this case, the mean of $X(t)$ is

$$E\{X(t)\} = \iint X f^\delta(X, t - t_0, \Xi) I(\Xi)\, d\Xi\, dX$$

The transformation $(\Xi, X) \to (\Xi, \Lambda)$, such that $X = Z(t - t_0, \Lambda, \Xi)$, allows us to write the preceding integral as the mean of the process Z, calculated with the joint density $F(\Xi, \Lambda) = I(\Xi)\rho(\Lambda)$ for the independent variables Ξ and Λ. The same is true for the variance matrix.

To picture the method which has just been exposed, we are going to calculate explicitly a process Z. For simplicity, we limit ourselves to an example in $\mathbb{R}$.

3. EXAMPLE IN $\mathbb{R}$

If $X(t) = x(t) \in \mathbb{R}$ is the variable of position, Eq. (1) is reduced to an equation of the Smoluchowski type. For $D = ax + b$ and $[M] = \mu$, it is written

$$\frac{\partial}{\partial t} f(x, t) = -\frac{\partial}{\partial x}\{(ax + b) f(x, t)\} + \mu \frac{\partial^2}{\partial x^2} f(x, t)$$

and its fundamental solution is

$$f^{\delta}(x, t - t_0, x_0) = \{2\pi A(t - t_0)\}^{-1/2} \exp -\frac{(x - x_t^D)^2}{2A(t - t_0)}$$

with

$$A(t - t_0) = \frac{\mu}{a}(e^{2a(t-t_0)} - 1), \qquad x_t^D = x_0 e^{a(t-t_0)} + \frac{b}{a}(e^{a(t-t_0)} - 1)$$

Hence we deduce that x_t^D and $A(t - t_0)$ are, respectively, the mean and the variance of the process $x(t)$. Let us put this solution into the dynamic equation (2); we obtain the differential equation

$$\frac{dx_t}{dt} = ax_t + b + \frac{\mu}{A(t - t_0)}(x_t - x_t^D)$$

whose general solution is

$$x_t = x_t^D + \lambda[A(t - t_0)]^{1/2}; \qquad \lambda \text{ a constant of integration}$$

We notice that the A function vanishes for $t = t_0$, and thus x_t and x_t^D correspond to the same initial condition $(x_0 t_0)$.

The Jacobian $J_{t'} = \partial x_{t'}/\partial\lambda = [A(t' - t_0)]^{1/2}$ is different from zero for $t' > t_0$. Given a final condition $x_{t'} = x'$ at time t', we then determine the value of the constant λ associated with the point $(x't')$:

$$\lambda = (x' - x_{t'}^D)/[A(t' - t_0)]^{1/2}$$

We thus obtain a sole trajectory going from $(x_0 t_0)$ to $(x't')$:

$$x_t = x_t^D + \frac{x' - x_{t'}^D}{[A(t' - t_0)]^{1/2}}[A(t - t_0)]^{1/2}; \qquad t' > t > t_0$$

Let us associate with this trajectory the probability density for arrival at x' at time t'. It is then possible to calculate the mean and the variance of x_t by integration over x', but it is better to deduce for λ a probability density ρ defined by $\rho(\lambda, t' - t_0, x_0)\, d\lambda = f^{\delta}(x', t' - t_0, x_0)\, dx'$. Because of the expressions which have been found for λ and $J_{t'}$, it appears that ρ is a Gaussian function only of λ:

$$\rho(\lambda) = (2\pi)^{-1/2} e^{-\lambda^2/2}$$

The set of the trajectories x_t starting from $(x_0 t_0)$ and corresponding to all the possible $(x't')$ points can then be considered as a process $z(t - t_0, \lambda, x_0)$:

$$z(t - t_0, \lambda, x_0) = x_t^D(t - t_0, x_0) + \lambda[A(t - t_0)]^{1/2}; \qquad \rho(\lambda) = (2\pi)^{-1/2} e^{-\lambda^2/2}$$

We immediately check that the mean and the variance of this process z are respectively equal to the mean and the variance of the process $x(t)$:

$$\int z(t - t_0, \lambda, x_0)\rho(\lambda)\, d\lambda = x_t^D(t - t_0, x_0)$$

$$\int \{z(t - t_0, \lambda, x_0) - x_t^D(t - t_0, x_0)\}^2 \rho(\lambda)\, d\lambda = A(t - t_0)$$

Let us consider now the case of any (normalized) initial density $I(\xi) \neq \delta(\xi - x_0)$. The mean of the process $x(t)$ is then

$$E\{x(t)\} = \iint x\{2\pi A(t - t_0)\}^{-1/2} \times \left\{\exp -\frac{[x - x_t^D(t - t_0, \xi)]^2}{2A(t - t_0)}\right\} I(\xi)\, d\xi\, dx$$

The transformation $(\xi, x) \to (\xi, \lambda)$, such that $x = z(t - t_0, \lambda, \xi)$, leads to

$$E\{x(t)\} = \iint \{x_t^D(t - t_0, \xi) + \lambda[A(t - t_0)]^{1/2}\} \times (2\pi)^{-1/2} e^{-\lambda^2/2} I(\xi)\, d\xi\, d\lambda$$

In this formula, we recognize the mean of the process z calculated with the joint probability density for the two independent variables λ and ξ. The same is true for the variance. Moreover, due to the linearity of z with respect to λ and ξ, the calculation is achieved very easily:

$$E\{x(t)\} = x_t^D(t - t_0, \bar{\xi}); \qquad \bar{\xi} = \int \xi I(\xi)\, d\xi$$

$$E\{[x(t) - x_t^D(t - t_0, \bar{\xi})]^2\} = e^{2a(t-t_0)}\overline{(\xi - \bar{\xi})^2} + A(t - t_0)$$

It should be noted that the particular form of the process z (in which λ and ξ are separate and ρ is Gaussian) does not proceed from the simplicity of the given example. Indeed we still obtain this form (Λ and Ξ separate, ρ Gaussian) in more sophisticated linear cases [for example, $X(t)$ in phase space for any quadratic potential]. The results obtained above have therefore a rather broad meaning.

4. CONCLUDING REMARKS

In this paper, we have considered a Brownian particle described by a process $X(t)$ in a field D linear and independent of time. With the process $X(t)$, starting from $(X_0 t_0)$, we associate a description intermediate between the usual stochastic description (nondifferentiable trajectories)

$$dX(t) = D(X(t))\, dt + dW(t)$$

and the classical deterministic description

$$dX_t^D = D(X_t^D)\,dt$$

whose solution, for the initial condition $(X_0 t_0)$, is the mean of the previous process.

In this intermediate description, to any final point $(X't')$, $t' > t_0$, there corresponds a unique differentiable trajectory X_t going from $(X_0 t_0)$ to $(X't')$ and such that

$$dX_t/dt = D(X_t) - [M]\,\mathrm{grad}_{X_t} \log f^\delta(X_t, t - t_0, X_0)$$

To this trajectory we assign the probability density $f^\delta(X', t' - t_0, X_0)$ for arrival at $(X't')$. The set of the differentiable trajectories thus defined constitutes a process $(Z(t - t_0, \Lambda, X_0);\ \rho(\Lambda))$ whose mean and variance matrix are the same as those of the process $X(t)$ when $(X_0 t_0)$ is given. In the general case, when X_0 is a random variable denoted Ξ, with density $I(\Xi)$, the mean and the variance matrix of the process $Z(t - t_0, \Lambda, \Xi)$, calculated with the joint density $I(\Xi)\rho(\Lambda)$, are still the same as those of the process $X(t)$.

Thus, to the real Brownian particles, we associate quasiparticles whose behavior presents both a certain deterministic aspect (in such a way that any given initial and final conditions determine a sole trajectory) and a probabilistic aspect (in such a way that a probability is attached to each of the trajectories). However, the process Z which describes the behavior of these quasiparticles is not equivalent to the process $X(t)$: it does not give back the covariance of $X(t)$ at two different instants. Yet it presents the advantage of simplicity. In particular, it allows us to achieve easily the calculation of the mean and of the variance matrix of the process $X(t)$ for any given initial density.

Finally, we note that the notion of Brownian quasiparticles introduced here is of some interest in the context of the recent stochastic interpretation of quantum mechanics (see, e.g., Refs. 2–6).

APPENDIX

In the case $D = 0$, the process $X(t)$ is reduced to the Wiener process $W(t)$ and for $n = 1$ we denote it, as in the text, $x(t)$. The associated Fokker–Planck equation is the heat equation, whose fundamental solution is

$$P(xt\,|x_0 t_0) = f^\delta(x, t - t_0, x_0) = \{4\pi(t - t_0)\mu\}^{-1/2} \exp -\frac{(x - x_0)^2}{4(t - t_0)\mu}$$

The mean and the variance of the process $x(t)$ are, respectively, x_0 and $2(t - t_0)\mu$. The process Y, which is introduced in the text, is written here

$$y(t, x', t', x_0, t_0) = \int x\, \frac{P(xt\,|x_0 t_0) P(x't'\,|xt, x_0 t_0)}{P(x't'\,|x_0 t_0)}\, dx; \qquad t' > t > t_0$$

where $P(x't'|xt, x_0t_0) = P(x't'|xt) = f^{\delta}(x', t' - t, x)$ because of the Markovian and homogeneous nature of the process $x(t)$. Thus

$$y(t, x', t', x_0, t_0) = \frac{(t - t_0)x' + (t' - t)x_0}{t' - t_0}$$

The mean of the process y being equal to x_0, its variance is

$$E\{(y - x_0)^2\} = \int \left(\frac{(t - t_0)(x' - x_0)}{t' - t_0}\right)^2 f^{\delta}(x', t' - t_0, x_0)\, dx' = 2\mu \frac{(t - t_0)^2}{t' - t_0}$$

As stated in the text, the variance of the process y at time t is different from the variance $2(t - t_0)\mu$ of the process $x(t)$ at the same time.

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